・ 以下，$c_1, c_2 ... C$ は定数とする．

・ $\ln()$ とは，$\log_e()$ のことである．

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■ **問題 2-2 1.**

(1) $\dfrac{dy}{dx} = \alpha \dfrac{y}{x}$ ・・・①

$u = \dfrac{y}{x}$ とする．

$y = ux$ $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$ ・・・②

①，②より，$u + x\dfrac{du}{dx} = \alpha u$

$x\dfrac{du}{dx} = u(\alpha - 1)$ $\dfrac{du}{u} = (\alpha - 1)\dfrac{dx}{x}$

$\displaystyle\int \frac{du}{u} = \int (\alpha - 1)\frac{dx}{x}$ $\ln|u| = (\alpha - 1)\ln|x| + c_1$

$\ln|u| = \ln|Cx^{\alpha - 1}|$ $\dfrac{y}{x} = Cx^{\alpha - 1}$

$\therefore y = Cx^{\alpha}$

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1.

(2) $y' = \dfrac{x-y}{x+y} = \dfrac{1-y/x}{1+y/x}$

$u = y/x$ と置くと

$y = ux,\ y' = u + xu'$

$y' = u + x\dfrac{du}{dx} = \dfrac{1-u}{1+u}$

$x\dfrac{du}{dx} = \dfrac{1-u}{1+u} - u = \dfrac{-\left(u^2+2u-1\right)}{1+u}$

$\displaystyle\int \frac{1+u}{-\left(u^2+2u-1\right)}du = \int \frac{1}{x}dx$

$-\dfrac{1}{2}\log\left|u^2+2u-1\right| = \log|x| + C$

$\log\left|u^2+2u-1\right| = -2\log|x| + -2C$

$u^2+2u-1 = x^{-2} + e^{-2C}$

$x^2\left(u^2+2u-1\right) = C \qquad (C = e^{-2C})$

$u = y/x$ を代入して

$y^2 + 2xy - \left(x^2 + C\right) = 0$

$y = \dfrac{-2x \pm \sqrt{4x^2 + 4\left(x^2+C\right)}}{2}$

$= -x \pm \sqrt{2x^2 + C}$

$f(u) - u$ $\qquad$ $\dfrac{1}{2}\left(u - \dfrac{1}{u}\right) - u$

$-\displaystyle\int \frac{2u}{1+u^2}du = \ln|x| + c \qquad -\ln\left|1+u^2\right| = \ln|x| + c$

$1 + u^2 = cx^{-1} \qquad 1 + \left(\dfrac{y}{x}\right)^2 = c\dfrac{1}{x}$

$\therefore x^2 + y^2 = Cx$ あるいは, $y = \pm\sqrt{Cx - x^2}$

(3) $y' = \dfrac{y^2 - x^2}{2xy}$ $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{y}{x} - \dfrac{x}{y}\right)$

$u = \dfrac{y}{x}$ とする.

$y' = \dfrac{1}{2}\left(\dfrac{y}{x} - \dfrac{x}{y}\right) = f(u) \cdots ①$

いま, $\displaystyle\int \frac{du}{f(u) - u} = \ln|x| + c$ へ①を代入 $\displaystyle\int \frac{1}{\frac{1}{2}\left(u - \frac{1}{u}\right) - u} du = \ln|x| + c$

$\displaystyle -\int \frac{2u}{1+u^2} du = \ln|x| + c$ $-\ln|1 + u^2| = \ln|x| + c$

$1 + u^2 = cx^{-1}$ $1 + \left(\dfrac{y}{x}\right)^2 = c\dfrac{1}{x}$

$\therefore x^2 + y^2 = Cx$ あるいは, $y = \pm\sqrt{Cx - x^2}$

問題 **2-2 3.**

$$\frac{dy}{dx}=\frac{-x+y+2}{x+y+4}$$

$v=x+a,\quad w=y+b$　とおく.

$$x+y+4=(x+a)+(y+b)=x+y+(a+b)$$

$$-x+y+2=-(x+a)+(y+b)=-x+y+(-a+b)$$

$$\begin{cases}a+b=4\\-a+b=2\end{cases}\qquad\therefore a=1,\quad b=3$$

$$v=x+1,\quad w=y+3$$

$$\frac{dw}{dv}=\frac{-v+w}{v+w}=\frac{-1+\dfrac{w}{v}}{1+\dfrac{w}{v}}\quad\cdots①$$

$u=\dfrac{w}{v}$ とする.

$$w=uv\qquad\frac{dw}{dv}=u+v\frac{du}{dv}\quad\cdots②$$

①, ②より, $u+v\dfrac{du}{dv}=\dfrac{-1+u}{1+u}$

$$v\frac{du}{dv}=\frac{-(1+u^2)}{1+u}\qquad\frac{-(1+u)}{1+u^2}du=\frac{dv}{v}$$

$$-\left\{\int\frac{1}{1+u^2}du+\int\frac{u}{1+u^2}du\right\}=\int\frac{1}{v}dv$$

$$\tan^{-1}u+\frac{1}{2}\ln\left|1+u^2\right|=-\ln|v|+c_1$$

$$2\tan^{-1}u+\ln\left|1+u^2\right|=-2\ln|v|+C$$

$$2\tan^{-1}u+\ln\left|(1+u^2)v^2\right|=C$$

$$\therefore 2\tan^{-1}\left(\frac{y+3}{x+1}\right)+\ln\left\{(x+1)^2+(y+3)^2\right\}=C$$

■問題 2-3　1.

(1)　$y'-\lambda y = ae^{\mu x}$

$y'-\lambda y = 0$　のとき，

$\dfrac{dy}{dx} = \lambda y$　　$\displaystyle\int \frac{1}{y}dy = \lambda\int dx$　　$\ln|y| = \lambda x + c$　　$\therefore y = Ce^{\lambda x}$

$z(x) = e^{\lambda x}$,　　$a(x) = C$　　とおく．

$y = a(x)z(x)$　　$y' = a'z + az' = a'z + a(\lambda z) = a'z + \lambda y$

$y'-\lambda y = ae^{\mu x}$　より，$a'z = ae^{\mu x}$　　$\dfrac{da}{dx}e^{\lambda x} = ae^{\mu x}$

$\displaystyle\int da = a\int \frac{e^{\mu x}}{e^{\lambda x}}dx$　・・①　$\displaystyle a(x) = a\int e^{(\mu-\lambda)x}dx = \frac{a}{\mu-\lambda}e^{(\mu-\lambda)x} + c$

$y = a(x)z(x)$　より，

$\displaystyle y = Ce^{\lambda x} + \frac{a}{\mu-\lambda}e^{(\mu-\lambda)x}e^{\lambda x}$　　$\displaystyle\therefore y = Ce^{\lambda x} + \frac{a}{\mu-\lambda}e^{\mu x}$

ただし，$\mu \neq \lambda$

$\mu = \lambda$ のとき，①式において

$a(x) = a\displaystyle\int dx$

$\Leftrightarrow a(x) = ax + c$

よって，　$y = (ax + c)e^{\lambda x}$　　$(\mu = \lambda)$

(2) $y'+xy=ax^3\cdots①$

$y'+xy=0$ のとき,

$$\int\frac{1}{y}dy=-\int xdx \quad \ln|y|=-\frac{1}{2}x^2+c \qquad \therefore y=Ce^{-x^2/2}$$

$Z(x)=e^{-x^2/2}$, $A(x)=C$ とおく.

$y=A(x)Z(x)$ $y'=A'Z+AZ'=A'Z-xAZ=A'Z-xy$

① と見比べて, $A'Z=ax^3$ $\dfrac{dA}{dx}=\dfrac{ax^3}{Z}$

$$A=\int\frac{ax^3}{e^{-x^2/2}}dx=a\int x^3e^{x^2/2}dx=a\int x^2(e^{x^2/2})'dx=a(x^2e^{x^2/2}-\int 2xe^{x^2/2}dx)$$

$$=a(x^2e^{x^2/2}-2e^{x^2/2})+c$$

$$\therefore y=Ce^{-x^2/2}+e^{-x^2/2}a(x^2e^{x^2/2}-2e^{x^2/2})=Ce^{-x^2/2}+a(x^2-2)$$

(3) $(1+x^2)y'+xy\sqrt{1+x^2}=0$

$$y'=-\frac{xy\sqrt{1+x^2}}{1+x^2}=-\frac{xy}{\sqrt{1+x^2}}$$

$$\int\frac{1}{y}dy=-\int\frac{x}{\sqrt{1+x^2}}dx$$

$$\log|y|=-\sqrt{1+x^2}+C$$

$$y=Ce^{-\sqrt{1+x^2}}$$

(4) $y' + (\cos x)y = \sin 2x$

$y' + (\cos x)y = 0$

$\int \frac{1}{y} dy = -\int \cos x dx$

$\log|y| = -\sin x + C$

$y = Ce^{-\sin x}$

$z(x) = e^{-\sin x}$, $a(x) = C$ と置く

$y = a(x)z(x)$

$y' = a'z + az'$

$= a'z + a(-\cos x)z$

$= a'z + (-\cos x)y$

$y' + (\cos x)y = \sin 2x$ より

$a'z = \sin 2x$

$\frac{da}{dx} e^{-\sin x} = \sin 2x$

$\int da = \int e^{\sin x}(\sin 2x)dx$

$a(x) = \int e^{\sin x}(2\sin x \cos x)dx$

$= 2\int e^{\sin x} \sin x \cos x dx$

$= 2\sin x e^{\sin x} - 2\int \cos x e^{\sin x} dx$

$= 2\sin x e^{\sin x} - 2e^{\sin x} + C$

$y = a(x)z(x)$

$= (2\sin x e^{\sin x} - 2e^{\sin x} + C)e^{-\sin x}$

$= 2\sin x - 2 + Ce^{-\sin x}$

$= Ce^{-\sin x} + 2(\sin x - 1)$